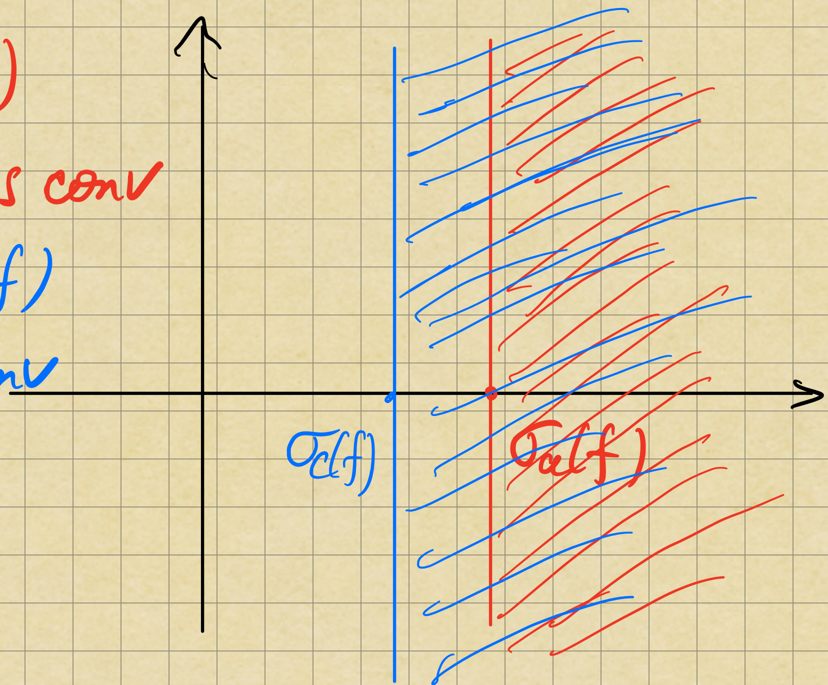
Recall: $f \in \mathcal{A} \rightsquigarrow L_f(s) := \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$.

$L_f(s)$ absolutely convergent for $Re(s) > \sigma_a(f)$
(and NOT abs conv for $Re(s) < \sigma_a(f)$).

<u>Theorem</u>: Let $f \in \mathcal{A}$. There exists $\sigma_c(f) \in \mathbb{R} \cup \{\pm\infty\}$ (called abscissa of conditional convergence) such that $L_f(s)$ converges if $\sigma > \sigma_c(f)$ and $L_f(s)$ does not converge if $\sigma < \sigma_c(f)$. The convergence is uniform on compact sets and $\sigma_a(f) - 1 \leq \sigma_c(f) \leq \sigma_a(f)$.

$Re(s) > \sigma_a(f)$
$\Rightarrow L_f(s)$ abs conv
$Re(s) > \sigma_c(f)$
$\Rightarrow L_f(s)$ conv



<u>Proof</u>: If $\{s \in \mathbb{C} : L_f(s) \text{ convergent}\} = \emptyset$, set $\sigma_c(f) = \infty$ and nothing to prove.

Otherwise, set $\sigma_c(f) := \inf\{\operatorname{Re}(s) : L_f(s) \text{ converges}\}$.

Let $K = [A, B] \times [C, D]$ a compact rectangle inside $\{s \in \mathbb{C} : \operatorname{Re}(s) > \sigma_c(f)\}$. Note that every compact set inside $\{\operatorname{Re}(s) > \sigma_c(f)\}$ is contained in such rectangle. Therefore it is enough to prove uniform convergence on K.

By definition, $\exists\, s_0 \in \{\sigma_c(f) \le \operatorname{Re}(s) < A : L_f(s) \text{ converges}\}$

Let $s \in K$. Define $S(s, y, x) := \sum_{y < n \le x} \frac{f(n)}{n^s}$

$$S_0(y, x) := \sum_{y < n \le x} \frac{f(n)}{n^{s_0}}.$$

Fix $\varepsilon > 0$. As $L_f(s_0)$ converges, $\exists\, Y_0 > 1$ such that $\forall\, x \ge y \ge Y_0$, we have $|S_0(y, x)| \le \varepsilon$. (by Cauchy criterion)

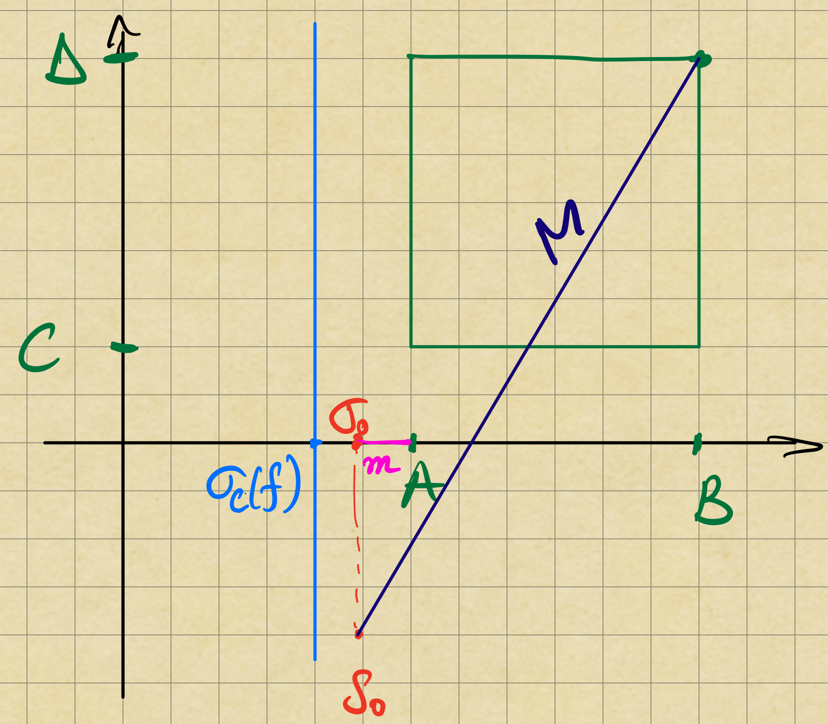
By partial summation,

$$S(s, y, x) = \sum_{y < n \le x} \frac{f(n)}{n^{s_0}} \cdot n^{s_0 - s} =$$

$$= S_0(y,x)\cdot x^{s_0-s} - (s_0-s)\int_y^x S_0(y,\xi)\,\xi^{s_0-s-1}\,d\xi$$

Set $m := \min\{\sigma-\sigma_0 : s\in K\} = A-\sigma_0 > 0.$

$M := \max\{|s-s_0| : s\in K\} > 0.$



$$|x^{s_0-s}| = x^{\sigma_0-\sigma} \le x^{-m}$$

$$\int_y^x |\xi^{s_0-s-1}|\,d\xi = \int_y^x \xi^{\sigma_0-\sigma-1}\,d\xi = \left[\frac{\xi^{\sigma_0-\sigma}}{\sigma_0-\sigma}\right]_y^x \le \frac{y^{-m}}{m}.$$

Hence for all $x > y \ge y_0$ and $s\in K$,

$$|S(s,y,x)| \le \varepsilon\cdot x^{-m} + \varepsilon\cdot\frac{M}{m}\,y^{-m} \le \varepsilon\left(1+\frac{M}{m}\right).$$

This shows $\sum_{n=1}^{N} \frac{f(n)}{n^s}$ is Cauchy and uniformly

convergent inside K.

It remains to show $\sigma_a(f) - 1 \leq \sigma_c(f) \leq \sigma_a(f)$.

RHS is clear.

Let $s \in \mathbb{C}$ such that $\sigma = Re(s) > \sigma_c(f)$. We want to show $\sigma + 1 > \sigma_a(f)$.

Let $s' \in \mathbb{C}$ s.t. $\sigma > \sigma' > \sigma_c(f)$ and $L_f(s')$ conv

Set $\delta := \sigma - \sigma'$.

$L_f(s')$ convergent $\Rightarrow \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $\left| \frac{f(n)}{n^{s'}} \right| \leq 1$.

$$\Rightarrow \sum_{n \geq N_0} \left| \frac{f(n)}{n^{s+1}} \right| = \sum_{n \geq N_0} \frac{|f(n)|}{n^{\sigma+1}} = \sum_{n \geq N_0} \frac{|f(n)|}{n^{\sigma'}} \cdot \frac{1}{n^{1+\delta}}$$

$$\leq \sum_{n \geq N_0} \frac{1}{n^{1+\delta}} < \infty.$$

$\Rightarrow L_f(s+1)$ absolutely convergent. $\square$

Holomorphicity of Dirichlet series in half-plane of conditional convergence

Recall the following theorem from complex analysis:

<u>Thm (Weierstrass)</u>:
Let $\Omega \subseteq \mathbb{C}$ open, $f_n : \Omega \to \mathbb{C}$ sequence of holomorphic functions on Ω. Assume that pointwise limit $f = \lim_{n\to\infty} f_n$ exists and that convergence is uniform on compacta.
Then f is holomorphic and moreover $f_n' \xrightarrow{n\to\infty} f'$ uniformly on compact sets.

<u>Corollary</u>: Let $f \in \mathcal{A}$. Then L_f is holomorphic on $\{ s \in \mathbb{C} : Re(s) > \sigma_c(f) \}$ and

$$L_f'(s) = \sum_{n=1}^{\infty} - \frac{f(n) \log n}{n^s} \quad \text{on } Re(s) > \sigma_c(f)$$

<u>Proof</u>: Let $f_N(s) := \sum_{n=1}^{N} \frac{f(n)}{n^s}$.

This is holomorphic on $\{s \in \mathbb{C} : Re(s) > \sigma_c(f)\}$ and $f_N \to f$ uniformly on compacta inside $\Omega = \{s \in \mathbb{C} : Re(s) > \sigma_c(f)\}$.

Also clearly $f_N'(s) = -\sum_{n=1}^{N} \frac{f(n) \log n}{n^s}$. $\boxed{}$

Algebraic properties of L-functions

Thm: Suppose $f, g \in \mathcal{A}$ and L_f, L_g both converge absolutely at s. Then L_{f*g} converges absolutely at s and $L_{f*g}(s) = L_f(s) L_g(s)$.

Proof: $$\sum_{n \geq 1} \left| \frac{f*g(n)}{n^s} \right| \leq \sum_{n \in \mathbb{N}} \sum_{n_1 n_2 = n} \frac{|f(n_1) g(n_2)|}{n^\sigma}$$

$$\leq \sum_{n \in \mathbb{N}} \sum_{n_1 n_2 = n} \left| \frac{f(n_1)}{n_1^s} \right| \left| \frac{g(n_2)}{n_2^s} \right|$$

$$= \left(\sum_{n_1 \in \mathbb{N}} \left| \frac{f(n_1)}{n_1^s} \right| \right) \left(\sum_{n_2 \in \mathbb{N}} \left| \frac{g(n_2)}{n_2^s} \right| \right) < \infty.$$

hence $L_{f*g}(s)$ absolutely convergent at s if $L_f(s)$ & $L_g(s)$ are abs conv at s.

Moreover, note that

$$L_{f*g}(s) = \sum_{n\in\mathbb{N}} \sum_{n_1 n_2 = n} \frac{f(n_1)}{n_1^s} \frac{g(n_2)}{n_2^s} = L_f(s) L_g(s)$$

in the region of absolute convergence. $\boxtimes$

Corollary: $\sigma_a(f*g) \le \max(\sigma_a(f), \sigma_a(g))$.

Thm: (Identity theorem).

Suppose $f, g \in \mathcal{A}$ and $\max\{\sigma_a(f), \sigma_a(g)\} < \sigma_0 < \infty$.

Suppose that $L_f(s) = L_g(s)$ for $Re(s) \ge \sigma_0$.

Then $f = g$.

Proof: Let $h = f - g \in \mathcal{A}$.

Then L_h conv abs for $Re(s) > \sigma_0$ and $L_h(s) = 0$.

Let $n_0 = \inf\{n : h(n) \neq 0\}$.

If $n_0 \neq \infty$, we have $\frac{h(n_0)}{n_0^s} = -\sum_{n > n_0} \frac{h(n)}{n^s}$

(for $Re(s) \ge \sigma_0$).

Thus, $|h(n_0)| \le \sum_{n > n_0} |h(n)| \left(\frac{n_0}{n}\right)^\sigma$

$$= \sum_{n > n_0} |h(n)| \left(\frac{n_0}{n}\right)^{\sigma_0} \left(\frac{n_0}{n}\right)^{\sigma-\sigma_0}$$

$$\leq \sum_{n > n_0} |h(n)| \left(\frac{n_0}{n}\right)^{\sigma} \left(\frac{n_0}{n_0+1}\right)^{\sigma-\sigma_0}$$

$$= \underbrace{n_0^{\sigma_0}}_{\text{indep of } s} \underbrace{\left(\frac{n_0}{n_0+1}\right)^{\sigma-\sigma_0}}_{\to 0 \text{ as } \sigma \to \infty} \underbrace{\sum_{n > n_0} \frac{|h(n)|}{n^{\sigma_0}}}_{\text{indep of } s}$$

$\Longrightarrow h(n_0) = 0$, which is absurd. $\blacksquare$

Informally, $L_f(s)$ determines uniquely $f(n)$ for $\mathrm{Re}(s) > \sigma_a(f)$ and moreover ~~we~~ know $L_{f*g}(s) = L_f(s) L_g(s)$ uniquely identifies $f*g$ on $\mathrm{Re}(s) > \max\{\sigma_a(f), \sigma_a(g)\}$.

<u>Examples</u>:

- $\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = L_{\varepsilon}(s)$

  <u>Exercise</u>: $\sigma_a(\varepsilon) = \sigma_c(\varepsilon) = 1$

- $L_{\tau}(s) = L_{\varepsilon * \varepsilon}(s) = \zeta^2(s)$ conv abs for $\mathrm{Re}(s) > 1$

- $L_e(s) = 1$

- $L_\mu(s) = L_e(s) L_\zeta(s)^{-1} = \frac{1}{\zeta(s)}$, for $Re(s) > 1$.

Since L_μ holomorphic on $Re(s) > 1$, it follows that $\zeta(s) \neq 0$ for $Re(s) > 1$.

- $L_{\log}(s) = \sum_{n \in \mathbb{N}} \frac{\log n}{n^s} = -L_\zeta'(s) = -\zeta'(s)$, for $Re(s) > 1$.

- $L_\Lambda(s) = L_{\log * \mu}(s) = -\frac{\zeta'(s)}{\zeta(s)}$, $Re(s) > 1$.

- $L_{id}(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^{s-1}} = \zeta(s-1)$, for $Re(s) > 2$.

- $L_\varphi(s) = L_\mu(s) L_{id}(s) = \frac{\zeta(s-1)}{\zeta(s)}$, for $Re(s) > 2$.

Euler products

Theorem (Euler product)

Let $f \in \mathcal{A}$ multiplicative. If L_f converges absolutely at $s \in \mathbb{C}$, then

$$L_f(s) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{\ell \in \mathbb{N}} \frac{f(p^\ell)}{p^{\ell s}}\right),$$

where the product on the right converges absolutely.

Moreover, if f completely multiplicative, then

$$L_f(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

Note $1 + \sum_{\ell \in \mathbb{N}} \frac{f(p^\ell)}{p^{\ell s}} = \sum_{\ell=0}^{\infty} \left(\frac{f(p)}{p^s}\right)^\ell = \frac{1}{1 - \frac{f(p)}{p^s}}$

if f completely multiplicative.

Proof: Note that for each prime p, $\sum_{\ell=1}^{\infty} \frac{f(p^\ell)}{p^{\ell s}}$ converges absolutely as so does $L_f(s)$.

Let $P_N := \prod_{p \le N} \left(1 + \sum_{\ell=1}^{\infty} \frac{f(p^\ell)}{p^{\ell s}}\right)$.

Note that if $n = p_1^{l_1} \dots p_k^{l_k}$; then $\frac{f(n)}{n^s} = \frac{f(p_1^{l_1})}{p_1^{l_1 s}} \cdots \frac{f(p_k^{l_k})}{p_k^{l_k s}}$ since f multiplicative.

Denote $B(N) = \{ n \in \mathbb{N} : p \mid n \Rightarrow p < N \}$
(set of natural numbers where all prime factors are less than N)

Let $\{p < N\} = \{p_1, \dots, p_r\}$.

$$\text{Then } P_N = \sum_{l_1, \dots, l_r = 0}^{\infty} \frac{f(p_1^{l_1}) \cdots f(p_r^{l_r})}{p_1^{l_1 s} \cdots p_r^{l_r s}} = \sum_{n \in B(N)} \frac{f(n)}{n^s}$$

(we have used unique prime factorisation and that $\sum_{l_j=0}^{\infty} \frac{f(p_j^{l_j})}{p_j^{l_j s}}$ is absolutely convergent).

$$\text{Therefore } |P_N - L_f(s)| = \left| \sum_{n \in B(N)} \frac{f(n)}{n^s} - \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right|$$

$$\leq \sum_{n \notin B(N)} \left| \frac{f(n)}{n^s} \right| \leq \sum_{n \geq N} \left| \frac{f(n)}{n^s} \right| \xrightarrow[N \to \infty]{} 0.$$

We have used that $L_f(s)$ absolutely convergent at s and that $\{1, 2, \dots, N-1\} \subseteq B(N)$.

The product is absolutely convergent since similarly

$$\prod_{p<N}\left(1+\left|\sum_{l=1}^{\infty}\frac{f(p^l)}{p^{ls}}\right|\right) \le \prod_{p<N}\left(1+\sum_{l=1}^{\infty}\left|\frac{f(p^l)}{p^{ls}}\right|\right) = \sum_{n\in B(N)}\left|\frac{f(n)}{n^s}\right| \le \sum_{n\in\mathbb{N}}\left|\frac{f(n)}{n^s}\right| < \infty,$$

for all N. $\boxtimes$

<u>Examples</u>: • $\zeta(s) = L_\varepsilon(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}$, for $\mathrm{Re}(s)>1$.

• $L_\tau(s) = \prod_p\left(1+\sum_{l=1}^{\infty}\frac{\tau(p^l)}{p^{ls}}\right) = \prod_p\left(1+\sum_{l=1}^{\infty}\frac{(l+1)}{p^{ls}}\right)$

Note that for $0<|x|<1$, $\frac{1}{1-x} = \sum_{n\geq 0} x^n$

Differentiating both sides, $\left(\frac{1}{1-x}\right)^2 = \sum_{n\geq 1} n x^{n-1}$.

Put $x = p^{-s}$, get $1+\sum_{l=1}^{\infty}\frac{(l+1)}{p^{ls}} = \left(1-\frac{1}{p^s}\right)^{-2}$, for $\mathrm{Re}(s)>1$.

Hence $L_\tau(s) = \zeta(s)^2$, which we already knew from $\tau = \varepsilon * \varepsilon$.

• $L_\mu(s) = \prod_p\left(1-\frac{1}{p^s}\right) = \zeta(s)^{-1}$, $\mathrm{Re}(s)>1$.